MISSION LOGIC DEVELOPMENT AND

TRAJECTORY PREDICTION, DETERMINATION AND OPTIMIZATION

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Introduction and Summary

Analytical Mechanics Associates, Inc., in association with the Mission Planning and Analysis Division of NASA Manned Spacecraft Center, has conducted a continuing technical effort in the field of orbit determination from observational data, midcourse guidance analysis, rocket trajectory optimization, and mission planning under Contract NAS 9-2527. The present report represents the final report under this contract and is submitted in fulfillment of contract reporting requirements. Since previous reports submitted have provided comprehensive coverage of the efforts in orbit determination, rocket trajectory optimization, and the development of a mission planning iterator, the present report will be devoted entirely to midcourse guidance analysis and the development of a simplified Encke method suitable for mission analysis purposes.

In the midcourse guidance analysis, equations were generated for propagating second order perturbation statistics when measurement data is received at a sequence of specified times. Linear feedback control, either continuous or as discrete impulses, was included. With the equations presented, the covariance matrices of perturbations from the nominal system path and of the estimation errors may be calculated. Included is a variant of the Min H optimization procedure which may be used to determine a velocity correction schedule which minimizes a statistical measure of the terminal miss and the fuel used.

A modification in the Encke method trajectory computation scheme has been developed and incorporated into the mission analysis program. An anomaly, β, rather than time, t, is used as the independent variable. With this change, Kepler's equations are no longer transcendental and an iterative solution procedure is unnecessary. Furthermore, geometric stopping and printing conditions are often expressed conveniently in terms of β when they would require iterative determinations of t. Testing has shown that savings in computer time are as high as 50% or more.

MIDCOURSE GUIDANCE OPTIMIZATION WITH DISCRETE MEASUREMENTS

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Introduction

Midcourse guidance in space missions is necessary to compensate for injection errors. A powerful technique for choosing a guidance law is based on statistical analysis of the perturbations from a pre-planned (nominal) path. The "initial" conditions for the analysis are the statistics of the injection errors. The analysis assumes that measurements are made along the path so that the state (position and velocity) of the system may be estimated. Velocity corrections are made to reduce predicted terminal errors. Imperfect implementation of the corrections, as well as imperfect knowledge of the state, makes it impossible to exactly satisfy terminal constraints. The chosen guidance law is judged according to statistical measures of the terminal miss and the fuel used.

The development here will parallel that of [1], which in turn relies heavily on developments in [2], [3], [4]. Expressions for the performance index will be derived, and the Min H method of numerical optimization will be described. The approach is the same as in [1]; the primary difference is that the measurements are now assumed to be at discrete times rather than continuous. Both continuous and discrete feedback control will be treated.

The essential practical motivation for the discrete measurement development is the expectation of greater numerical accuracy. The limited numerical results obtained for continuous measurement theory show substantial inaccuracy in the estimation error covariance matrix, even for painfully small computing intervals. On theoretical grounds alone we would not expect a major difference between continuous measurements and closely spaced discrete measurements, if the measurement noise were appropriately scaled with the spacing interval.

One additional advantage may be gained if discrete velocity corrections are also assumed. Analytical approximations to the transition matrix between measurement and correction times may be made, thus eliminating the need for numerical integration of the covariance matrices altogether. All covariance matrices would be propagated via difference equations.

The Perturbation Equations

Each system path is assumed to involve only a "small" perturbation from a nominal path, small enough that all perturbation equations may be linearized. We may write

$$\delta\dot{x} = F\,\delta x + G(\delta u + w) \tag{1}$$

where

$\delta x(t)$ is the n-vector of state variable perturbations

$\delta u(t)$ is the m-vector of commanded feedback control

$w(t)$ is the m-vector of control implementation error

$F(t)$ and $G(t)$ are $n \times n$ and $n \times m$ matrices respectively, evaluated on the nominal path

$(\dot{\ }) = \frac{d}{dt}(\ )$

First we shall examine the continuous feedback problem, for which (1) is directly appropriate. To make the numerical analysis feasible, we assume linear feedback control (as in [1], [3], [4]). This means

$$\delta u(t) = -\Lambda(t)\,\delta\hat{x}(t) \tag{2}$$

where

$\delta\hat{x}(t)$ is the n-vector of estimates of δx

$\Lambda(t)$ is the $m \times n$ feedback gain matrix, to be chosen under some optimization criterion

Furthermore, as in [1], we assume that

$$\Lambda(t) = k(t)\,\Gamma(t) \tag{3}$$

where $\Gamma(t)$ is an $m \times n$ matrix evaluated along the nominal path and $k(t)$ is a scalar. $\Gamma(t)$ is determined so that $\delta u(t)$ will be a control which acts to reduce m specified linear functions of the predicted terminal state perturbations. In view of the linearized perturbation assumption, this is equivalent to reducing an m-component predicted terminal "miss", where a miss component may be any function of the terminal state and time. The remaining free choice, then, is the scalar $k(t)$.

The estimate of the perturbations is changed discontinuously at a measurement time, according to

$$\delta\hat{x}_{+} = \delta\hat{x}_{-} + K(z - M\,\delta\hat{x}_{-}) \tag{4}$$

where

$$z = \rho(x) - \rho(x_{nom}) + v \tag{5}$$

$\rho(x)$ is a p-vector of measurement functions

v is a p-vector of measurement errors

$M = \left(\frac{\partial\rho}{\partial x}\right)_{nom}$ is the $p \times n$ matrix of partials of p components with respect to x components

K is the $n \times p$ estimator gain matrix

Between measurement times the estimate is simply integrated from

$$\delta\dot{\hat{x}} = F\,\delta\hat{x} + G\,\delta u$$
$$= (F - kG\Gamma)\,\delta\hat{x} \qquad (6)$$

Secondly, we examine the discrete control problem. We assume that the control occurs only in very short intervals following measurements. Mathematically, we let the control become indefinitely large as the interval shrinks to zero, with a finite change of state. We write

$$\delta x_{+} = \delta x_{-} + G(\Delta_u + \Delta_w) \qquad (7)$$

where

$$\Delta_u = \int_{t_-}^{t_+} \delta u(\tau)\,d\tau \quad , \quad \Delta_w = \int_{t_-}^{t_+} w(\tau)\,d\tau$$

$\delta u(t)$ is still given by (2) and $\Lambda(t)$ by (3). Furthermore, $\Gamma(t)$ may be the same as with continuous control. The difference is that $k(t)$ becomes a Dirac function, with only $\int_{t_-}^{t_+} k(\tau)\,d\tau$ being important.

The velocity correction causes a change in $\delta\hat{x}$ equal to $G\Delta_u$. In our analysis, we shall assume that the sequence of measurement times t_i, $i = 1, 2, ---, N$ is prescribed. For the discrete control problem, we assume that a correction is permissible just following each measurement. For this reason we adopt the subscript notation

$(\ )_-$ = ( ) just before the measurement

$(\ )_+$ = ( ) just after the measurement but before the impulsive control

$(\ )_{++}$ = ( ) just after the impulsive control

Thus, we replace (7) by the two equations

$$\delta x_{+} = \delta x_{-} \tag{8}$$

$$\delta x_{++} = \delta x_{+} + G(\Delta_u + \Delta_w) \tag{9}$$

For measurement and correction time t_n, we write

$$k_n = \int_{t_{n-}}^{t_{n+}} k(\tau)\, d\tau \tag{10}$$

The k_n become the parameters open for optimization.

The perturbation estimate obeys (4) and

$$\delta \hat{x}_{++} = \delta \hat{x}_{+} + G\,\Delta_u \tag{11}$$

Covariance Matrix Equations

The second order statistics of the perturbations are described by covariance matrices. We choose, as in [1], the covariance matrices of the perturbation estimate $\delta\hat{x}$ and of the estimation error $e = \delta x - \delta\hat{x}$, defining

$$Y = \mathcal{E}[\delta\hat{x}\ \delta\hat{x}^T] \tag{12}$$

$$P = \mathcal{E}[e\ e^T] \tag{13}$$

These matrices are sufficient for the description because $\mathcal{E}[\delta\hat{x}\ e^T]$ turns out to be zero.

First let us consider the continuous feedback control problem. Between measurements, (1) and (6) apply. The propagation of P depends on the statistics of w. Subtracting (6) from (1) gives

$$\dot{e} = F e + G w \tag{14}$$

We assume that $w(t)$ is zero mean white noise with

$$\mathcal{E}[w(t)\ w^T(\tau)] = Q(t)\ \delta(t - \tau) \tag{15}$$

Then

$$\dot{P} = F P + P F^T + G Q G^T \tag{16}$$

The propagation of Y follows directly from (6):

$$\dot{Y} = (F - kG\Gamma)Y + Y(F - kG\Gamma)^T \tag{17}$$

At a measurement time the estimation error obeys

$$e_+ = (I - KM)e_- + Kv \tag{18}$$

This follows from continuity of δx and from (4) if z is expanded in terms of δx and only terms through linear are retained. We assume that v is a zero mean error, uncorrelated between measurement times. The covariance is given by

$$\mathcal{E}[v(t_i)\, v^T(t_j)] = R(t_i)\, \delta_{ij} \tag{19}$$

Using (18) and (19) we immediately obtain

$$P_+ = (I - KM)P_- + P_-(I - KM)^T + KRK^T \tag{20}$$

The minimum variance estimator gain satisfies the now well known expression

$$K = P_- M^T (M P_- M^T + R)^{-1} \tag{21}$$

Substituting (21) into (20) gives

$$P_+ = P_- - P_- M^T (M P_- M^T + R)^{-1} M P_- \tag{22}$$

Since $\mathcal{E}[\delta x\ \delta x^T] = Y + P$ is continuous across a measurement,* it follows immediately that Y is discontinuous according to

$$Y_+ = Y_- - P_+ + P_- \tag{23}$$

---

* This equality requires the vanishing of $\mathcal{E}[\delta\hat{x}\ e^T]$. Between measurement times $\frac{d}{dt}\mathcal{E}[\delta\hat{x}\ e^T]$ is homogeneous in $\mathcal{E}[\delta\hat{x}\ e^T]$. The inhomogeneous term in the discontinuity at a measurement time vanishes due to (21). With $\mathcal{E}[\delta\hat{x}\ e^T] = 0$ at $t = t_0$, it will equal zero throughout.

Equations (16), (17), (22) and (23) suffice to propagate P and Y if $Q(t)$ and R are known. We assume $R_n = R(t_n)$ is given for every measurement time. The implementation error covariance matrix, however, we take as obeying

$$
\begin{aligned}
Q(t) &= \sigma^2 \mathcal{E}[\delta u \; \delta u^T] + S(k)\bar{Q} \\
&= \sigma^2 k^2 \Gamma Y \Gamma^T + S(k)\bar{Q}
\end{aligned} \tag{24}
$$

where σ^2 and $\bar{Q}$ are given constants, and $S(k)$ is the unit step function. In rough statistical terms, the implementation error consists of one term proportional to the control magnitude and one, when the control is non-zero, independent of the control. This covariance matrix is thought to be a reasonable representation for thrust errors of a conventional rocket engine.

Next we consider the approximation of impulsive velocity correction. We observe that in the absence of measurements or control between the discrete times t_n, P and Y obey the same differential equation. In (16) Q is zero, and in (17) k is zero between the times, so that

$$\dot{P} = FP + PF^T \tag{25}$$

$$\dot{Y} = FY + YF^T \tag{26}$$

Thus, if the transition matrix $\Phi_{n,n-1}$ is available, we may write

$$P_{n-} = \Phi_{n,n-1} P_{(n-1)++} \Phi^T_{n,n-1} \tag{27}$$

$$Y_{n-} = \Phi_{n,n-1} Y_{(n-1)++} \Phi^T_{n,n-1} \tag{28}$$

Across the measurement, equations (22) and (23) still hold; the data processing procedure is identical with the first case processing. Immediately following the measurement a velocity correction is made, described by (9) and (11). To calculate the covariance changes we must evaluate $\mathcal{E}[\Delta_u \Delta_u^T]$ and $\mathcal{E}[\Delta_w \Delta_w^T]$.

$$\Delta_u(t_n) = \int_{t_{n-}}^{t_{n+}} \delta u(\tau)\, d\tau$$

$$= - \Gamma(t_n)\, \delta\hat{x}_{n+} \int_{t_-}^{t_+} k(\tau)\, d\tau$$

$$= - k_n \Gamma_n\, \delta\hat{x}_{n+} \tag{29}$$

Hence

$$\mathcal{E}[\Delta_u \Delta_u^T]_n = k_n^2 \Gamma_n Y_{n+} \Gamma_n^T \tag{30}$$

The implementation error covariance matrix we choose by exact analogy to (24)

$$\mathcal{E}[\Delta_w \Delta_w^T]_n = Q_n = \sigma^2 \mathcal{E}[\Delta_u \Delta_u^T]_n + S(k_n)\bar{Q}_n \tag{31}$$

σ is the same number as that used in the continuous feedback case, but $\bar{Q}_n$ differs from $\bar{Q}$ by a time factor. $\bar{Q}_n$ may be interpreted as the mean square error in the velocity correction due to timing errors in switching the engine on, then off. In fact, $\bar{Q}$ might be chosen as $\bar{Q}_n$ divided by the length of time the engine was on in the neighborhood of t_n.

Using (29), (30) and (31) along with (9) and (11), we find that

$$Y_{n++} = Y_{n+} - k_n G\Gamma Y_{n+} - k_n Y_{n+}\Gamma^T G^T + k_n^2 G\Gamma Y_{n+}\Gamma^T G^T \tag{32}$$

$$P_{n++} = P_{n+} + GQ_n G^T \tag{33}$$

The Performance Index

There are usually two conflicting aims in midcourse guidance: keeping the terminal errors small and avoiding excessive fuel requirements. There is no universally agreed upon "best" way to separate the midcourse guidance optimization problem from the overall mission success probability problem. The performance measure chosen here was used in [3] for an idealized but very useful optimization analysis, and also in the analyses of [1] and [4] for the midcourse problem with continuous measurements. The performance index is taken to be

$$J_* = \mathcal{E}[\delta x^T L_x \delta x]_{t=t_f} + c \int_{t_o}^{t_f} \mathcal{E}\,|\delta u(\tau)|\, d\tau \tag{34}$$

The first term in J_* is simply a mean square miss. It could be the mean square position deviation from the nominal, or any other quadratic form in δx. The second term is proportional to the average of the fuel used, assuming that fuel flow is proportional to thrust magnitude. Unfortunately, there is no even moderately complex, let alone simple, way to express $\mathcal{E}|\delta u|$. Instead we determine its upper bound through

$$\mathcal{E}|\delta u| = \mathcal{E}\sqrt{\delta u^T\,\delta u} = \mathcal{E}\sqrt{\mathrm{tr}\,(\delta u\ \delta u^T)} \le \sqrt{\mathcal{E}\,\mathrm{tr}\,(\delta u\ \delta u^T)}$$

$$\sqrt{\mathcal{E}\,\mathrm{tr}\,(\delta u\ \delta u^T)} = \sqrt{\mathrm{tr}\ \mathcal{E}(\delta u\ \delta u^T)} \tag{35}$$

where tr stands for trace. Substituting (35) into (34), also using the trace operator in the first term of J_*, the amended performance index becomes

$$J = \operatorname{tr} L_x \,\mathcal{E}(\delta x\, \delta x^T)_{t=t_f} + c \int_{t_o}^{t_f} \sqrt{\operatorname{tr} \mathcal{E}(\delta u\, \delta u^T)}\; dt$$

$$= \operatorname{tr} [L_x (Y + P)]_f + c \int_{t_o}^{t_f} |k| \sqrt{\operatorname{tr} \Gamma Y \Gamma^T}\; dt \tag{36}$$

This expression serves both for continuous and discrete control, but it is clearer with discrete control to write J as

$$J = \operatorname{tr} [L_x (Y + P)]_f + c \sum_{i=1}^{N} |k_n| \sqrt{\operatorname{tr} \Gamma_n Y_{n+} \Gamma^T} \tag{37}$$

In both cases the constant c establishes the relative weighting between the miss and the fuel used. The control gains $k(t)$ or k_n will be smaller when c is larger. In fact, above some c the optimal gains will be zero. If $\mathcal{E}(\delta u^T \delta u)$ instead of $\mathcal{E}|\delta u|$ appeared in (34), k would be non-zero for any finite c.

The Min H Optimization Procedure

The optimization procedure to be described here is basically the same as that used in [1], with modifications made only to accommodate the discrete measurements and perhaps discrete control corrections. Although the derivation will appear somewhat different, the concepts and approximations are the same.

For the continuous control case, we define the Hamiltonian by

$$H = \operatorname{tr} L_p[FP + PF^T + GQG^T] + \operatorname{tr} L_y[(F - kG\Gamma)Y + Y(F - kG\Gamma)^T] + c|k|\sqrt{\operatorname{tr} \Gamma Y \Gamma^T} \tag{38}$$

With (38), (36) may be rewritten as

$$J = \operatorname{tr}[L_x(Y+P)]_f + \int_{t_o}^{t_{1-}} + \int_{t_{1+}}^{t_{2-}} + \cdots + \int_{t_{N+}}^{t_f} [H - \operatorname{tr} L_p \dot{P} - \operatorname{tr} L_y \dot{Y}]dt \tag{39}$$

For the optimal $k(t)$ there is no change in $k(t)$ which can decrease J. The approach is to write the expression for a small change, dJ, in J and try to establish that it is non-negative. We assume that t_f, the nominal terminal time, is fixed. Thus dJ is the variation δJ. This does not mean that every trajectory must terminate at t_f. For example, L_x may be chosen (see [4]) so that

$$\mathcal{E}[\delta x^T L_x \delta x]_f = \mathcal{E}[(a^T dx_f)^2] \tag{40}$$

Taking the variation of (39) (after substituting for Q) we obtain

$$\delta J = \operatorname{tr}[L_x(\delta Y + \delta P)]_f + \int_{t_o}^{t_{1-}} + \int_{t_{1+}}^{t_{2-}} + \cdots + \int_{t_{N+}}^{t_f} \{\operatorname{tr}(L_p F + F^T L_p)\,\delta P$$

$$+ \operatorname{tr}\left(\sigma^2 k^2 \Gamma^T G^T L_p G \Gamma - k L_y G \Gamma - k \Gamma^T G^T L_y + \frac{c|k|\,\Gamma^T \Gamma}{2\sqrt{\operatorname{tr} \Gamma Y \Gamma^T}}\right) \delta Y$$

$$+ H(k + \delta k) - H(k) - \operatorname{tr} L_p\, \delta \dot{P} - \operatorname{tr} L_y\, \delta \dot{Y}\}\,dt \tag{41}$$

The $\delta \dot{P}$ and $\delta \dot{Y}$ terms are integrated by parts. L_p and L_y are chosen to satisfy

$$\dot{L}_p + L_p F + F^T L_p = 0 \tag{42}$$

$$\dot{L}_y - k L_y G \Gamma - k \Gamma^T G^T L_y + \sigma^2 k^2 \Gamma^T G^T L_p G \Gamma + \frac{c|k|\,\Gamma^T \Gamma}{2\sqrt{\operatorname{tr} \Gamma Y \Gamma^T}} = 0 \tag{43}$$

Then, δJ may be expressed as

$$\delta J = \operatorname{tr}[L_x(\delta Y + \delta P)]_f - \sum_{n=1}^{N} \operatorname{tr}(L_p\, \delta P)_{n-} + \sum_{n=1}^{N} \operatorname{tr}(L_p\, \delta P)_{n+}$$

$$- \operatorname{tr}(L_p\, \delta P)_f + \operatorname{tr}(L_p\, \delta P)_o - \sum_{n=1}^{N} \operatorname{tr}(L_y\, \delta Y)_{n-} + \sum_{n=1}^{N} \operatorname{tr}(L_y\, \delta Y)_{n+}$$

$$- \operatorname{tr}(L_y\, \delta Y)_f + \operatorname{tr}(L_y\, \delta Y)_o + \int_{t_o}^{t_{1-}} + \int_{t_{1+}}^{t_{2-}} + \cdots + \int_{t_{N+}}^{t_f} [H(k + \delta k) - H(k)]\,dt \tag{44}$$

Since $P(t_o)$ and $Y(t_o)$ are assumed given, δP_o and δY_o are zero. The δP_f and δY_f terms are cancelled by choosing

$$L_p(t_f) = L_y(t_f) = L_x \tag{45}$$

Linearized perturbations of (22) and (23) satisfy

$$\begin{aligned}\delta P_+ = \delta P_- - \delta P_- M^T(MP_-M^T+R)^{-1}MP_- - P_-M^T(MP_-M^T+R)^{-1}M\,\delta P_- \\ + P_-M^T(MP_-M^T+R)^{-1}M\,\delta P_-M^T(MP_-M^T+R)^{-1}MP_-\end{aligned} \tag{46}$$

$$\delta Y_+ = \delta Y_- - \delta P_+ + \delta P_- \tag{47}$$

δP_+ and δY_+ in (44) are eliminated in favor of δP_- and δY_-. Then the coefficients of δP_- and δY_- (after suitable permutation of terms in each trace) are set equal to zero to give

$$\begin{aligned}L_{p-} = L_{p+} - M^T(MP_-M^T+R)^{-1}MP_-(L_p-L_y)_+ - (L_p-L_y)_+P_-M^T(MP_-M^T+R)^{-1}M \\ + M^T(MP_-M^T+R)^{-1}MP_-(L_p-L_y)_+P_-M^T(MP_-M^T+R)^{-1}M\end{aligned} \tag{48}$$

$$L_{y-} = L_{y+} \tag{49}$$

Now (44) collapses to

$$\delta J = \int_{t_o}^{t_{1-}} + \int_{t_{1+}}^{t_{2-}} + \cdots + \int_{t_{N+}}^{t_f} [H(k+\delta k) - H(k)]dt \tag{50}$$

Except for having the integral broken into pieces, the form of (50) is now the same as was found in [1]. The Min H method from [1] may now be applied to successively solve for $\delta k(t)$ histories. The difference here is that L_p and L_y, as well as P and Y, undergo discontinuities as given. The procedure is to choose $k(t)$, obtain $P(t)$ and $Y(t)$, then $L_p(t)$ and $L_y(t)$. $k(t) + \delta k(t)$ is then calculated just as in [1]. As a result of the other discontinuities $k(t)$ will become discontinuous at measurement times.

The entire procedure is only slightly different for the case of discrete control. There we define the Hamiltonian through

$$
\begin{aligned}
H_n = \operatorname{tr}[& L_{p_{n++}} G Q_n G^T - L_{p_{n+}} P_{n-} M^T (M P_{n-} M^T + R)^{-1} M P_{n-} \\
& + L_{p_{n-}} (\Phi P_{(n-1)++} \Phi^T - P_{(n-1)++}) + L_{y_{n++}} (-k_n G \Gamma Y_{n+} - k_n Y_{n+} \Gamma^T G^T \\
& + k_n^2 G \Gamma Y_{n+} \Gamma^T G^T) + L_{y_{n+}} P_{n-} M^T (M P_{n-} M^T + R)^{-1} M P_{n-} \\
& + L_{y_{n-}} (\Phi Y_{(n-1)++} \Phi^T - Y_{(n-1)++})] + \frac{|k_n|}{c} \sqrt{\operatorname{tr} \Gamma Y_{n+} \Gamma^T}
\end{aligned}
\tag{51}
$$

where G, Q, R, Γ, M are evaluated at t_n, $\Phi = \Phi_{n,n-1}$. Using this definition of H_n, J of (37) may be written as

$$
\begin{aligned}
J = \operatorname{tr}[L_x(Y+P)]_f + \sum_{n=1}^{N} [& H_n - \operatorname{tr} L_{p_{n++}} (P_{n++} - P_{n+}) - \operatorname{tr} L_{p_{n+}} (P_{n+} - P_{n-}) \\
& - \operatorname{tr} L_{p_{n-}} (P_{n-} - P_{(n-1)++}) - \operatorname{tr} L_{y_{n++}} (Y_{n++} - Y_{n+}) - \operatorname{tr} L_{y_{n+}} (Y_{n+} - Y_{n-}) \\
& - \operatorname{tr} L_{y_{n-}} (Y_{n-} - Y_{(n-1)++})]
\end{aligned}
\tag{52}
$$

A slight rearrangement of terms makes the derivation of the adjoint equations easier to see.

$$\sum_{n=1}^{N} \mathrm{tr}\, L_{p_{n++}} P_{n++} = - \mathrm{tr}\, L_{p_{N++}} P_{N++} + \mathrm{tr}\, L_{p_{o++}} P_{o++} + \sum_{n=1}^{N} L_{p_{(n-1)++}} P_{(n-1)++} \tag{53}$$

An exactly similar expression for Y is used. These substitutions, the substitution for Q_n, and the substitution $(Y+P)_f = \Phi_{fN} (Y+P)_{N++} \Phi_{fN}^T$ are made in J. The variation of J is formed. The coefficients of δY_{N++} and δP_{N++} set equal to zero give

$$L_{p_{N++}} = \Phi_{fN}^T L_x \Phi_{fN} \tag{54}$$

$$L_{y_{N++}} = \Phi_{fN}^T L_x \Phi_{fN} \tag{55}$$

The coefficients of $\delta P_{(n-1)++}$, δP_{n+}, δP_{n-}, $\delta Y_{(n-1)++}$, δY_{n+}, δY_{n-}, set equal to zero, give the equations for propagating L_p and L_y. They are

$$L_{p_{(n-1)++}} = \Phi_{n,n-1}^T L_{p_{n-}} \Phi_{n,n-1} \tag{56}$$

$$\begin{aligned} L_{p_{n-}} = {} & L_{p_{n+}} - (L_{p_{n+}} - L_{y_{n+}}) P_{n-} M^T (M P_{n-} M^T + R)^{-1} M \\ & - M^T (M P_{n-} M^T + R)^{-1} M P_{n-} (L_{p_{n+}} - L_{y_{n+}}) \\ & + M^T (M P_{n-} M^T + R)^{-1} M P_{n-} (L_{p_{n+}} - L_{y_{n+}}) P_{n-} M^T (M P_{n-} M^T + R)^{-1} M \end{aligned} \tag{57}$$

$$L_{p_{n+}} = L_{p_{n++}} \tag{58}$$

$$L_{y_{(n-1)++}} = \Phi^T_{n,n-1} L_{y_{n-}} \Phi_{n,n-1} \tag{59}$$

$$L_{y_{n-}} = L_{y_{n+}} \tag{60}$$

$$L_{y_{n+}} = L_{y_{n++}} - k_n L_{y_{n++}} G\Gamma - k_n \Gamma^T G^T L_{y_{n++}} + k_n^2 \Gamma^T G^T (L_{y_{n++}} + \sigma^2 L_{p_{n++}}) G\Gamma + \frac{c|k_n| \Gamma^T \Gamma}{2\sqrt{\mathrm{tr}\, \Gamma Y_{n+} \Gamma^T}} \tag{61}$$

The Hamiltonian H_n has the following terms involving k_n:

$$|k_n| c \sqrt{\mathrm{tr}\, \Gamma Y_{n+} \Gamma^T} - k_n \,\mathrm{tr}\,(L_{y_{n++}} G\Gamma Y_{n+} + Y_{n+} \Gamma^T G^T L_{y_{n++}}) + k_n^2 \,\mathrm{tr}\, \Gamma^T G^T (L_{y_{n++}} + \sigma^2 L_{p_{n++}}) G\Gamma + S(k_n)\, \mathrm{tr}\, L_{p_{n++}} G \bar{Q}_n G^T \tag{62}$$

k_n is assumed non-negative, so expression (62) may be written

$$a_n k_n + b_n k_n^2 + c_n S(k_n) \tag{63}$$

This is precisely the same form as obtained for continuous control. The k_n sequence for the next iteration is the one which minimizes (63), using

P, Y, L_p, L_y obtained with the current k_n sequence. Because it appears certain that the optimal k_n will satisfy $0 \leq k_n \leq 1$, these inequality constraints are imposed for each iteration.

The discrete control case has one potential numerical advantage. It may well happen that, after a "few" iterations, the times t_n, for which $k_n \neq 0$, will remain the same for further iterations. Knowing the times for non-zero corrections is usually considered more important than knowing with great precision the values of k_n at those times.